A NEW CURRENT REGURALIZATION OF THIRRING MODEL

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ABSTRACT

We study an ambiguity of the current regularization in the Thirring model. We find a new current definition which enables to make a comprehensive treatment of the current. Our formulation is simpler than Klaiber's formulation. We compare our result with other formulations and find a very good agreement with their result. We also obtain the Schwinger term and the general formula for any current regularization.

1 Introduction

The Thirring model has been investigated by many people. It is well-known that the Thirring model is an exactly solvable quantum field theory model in (1+1) dimensions [1]. An extensive investigation of the model was given by Hagen [2] and Klaiber [3]. Hagen introduced an external field and gave the general solution of the Thirring model. Klaiber analyzed the Thirring model and found the operator solutions which are expressed in terms of a free massless Dirac field. He constructed the solution to fulfill the positive definiteness. On the other hand, Nakanishi expressed the solution in terms of the free massless bosonic field [4]. He asserted that *all Heisenberg operators should be expressed in terms of asymptotic fields* from the standpoint of the general principle of quantum field theory. In the present paper, we use the bosonic expression (*bosonization*) [5, 6].

One of the methods for solving the quantum field theory is to determine the Operator Product Expansion (OPE) [7, 8]. In this formalism, the short distance

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behavior for products of the two local fields is important. For some case, we can determine the OPE exactly, e.g. (1+1)-dimensional Conformal Field Theory [9]. Concerning the operator products of the quantum field, there are difficulties with respect to the current regularization. In most cases, the current is defined by the limiting procedures as $\bar{\psi}(x+\epsilon)\gamma_\mu\psi(x)$. In the Thirring model, there are several definitions of the current. For example, the Schwinger current [10] is defined by limiting from spacelike direction only. The Johnson current [11] is defined by limiting from not only spacelike direction but also timelike direction symmetrically as

$$j^\mu(x) = \frac{1}{2}\Big[j^\mu(x;\,\epsilon) + j^\mu(x;\,\widetilde{\epsilon})\Big]_{\epsilon,\widetilde{\epsilon}\to 0}, \tag{1}$$

where ϵ and $\widetilde{\epsilon}$ are a timelike and spacelike vectors, respectively. Both current definitions are consistent with the solution of the Thirring model. However, the coupling constant is affected by the current definition. Therefore, in the Thirring model, the coupling constant is determined only when we define the current regularization [3]. It is also noted that these coupling constants are not independent. The coupling constant of the Schwinger definition g_S is given by

$$g_\mathrm{S} = \frac{g_\mathrm{J}}{1 - g_\mathrm{J}/2\pi}, \tag{2}$$

where g_J is the coupling constant of the Johnson definition. These current ambiguities also appear in the massive Thirring model, which we do not understand yet [12].

On the other hand, Dell'Antonio, Frishman and Zwanziger [14] analyzed the Thirring model without looking into the structure of the current. They extend the Johnson result [11]. They start with defining the commutation relations of the current, *current algebra formulation* [7, 8]. There are three parameters and we have two relations among them. Therefore, we can construct the current algebra from the lagrangian and the suitable definition of the current which has one parameter.

In this paper, we present a new current regularization of the Thirring model. We introduce one parameter in the definition. Our formulation is simpler than Klaiber's one and the new current definition is consistent with other formulations. The Thirring current and field can be written in terms of the free massless bosonic field. Therefore, we can analyze the model exactly.

In this paper, we employ the following notation:

$$x^\pm = x^0 \pm x^1,\ x_\pm = x^\mp/2,\ \partial_\pm = (\partial_0 \pm \partial_1)/2,\ \partial^\pm = 2\partial_\mp \tag{3}$$

and gamma matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

The anti-symmetric tensor $\epsilon_{\mu\nu}$ is taken to be $\epsilon_{10} = \epsilon^{01} = -1$.

2 Thirring model

The Thirring model is (1+1) dimensional field theory with the current-current interaction. The lagrangian of the Thirring model is given by

$$\mathcal{L} = \bar{\psi} \mathrm{i} \partial_\mu \gamma^\mu \psi - \frac{g}{2} j_\mu j^\mu, \qquad j_\mu = \bar{\psi} \gamma_\mu \psi, \tag{5}$$

where g is a coupling constant. Then, the equations of motion become

$$\mathrm{i}\partial_+ \psi_\mathrm{R} = g j_\mathrm{L} \psi_\mathrm{R}, \quad \mathrm{i}\partial_- \psi_\mathrm{L} = g j_\mathrm{R} \psi_\mathrm{L}, \tag{6}$$

where

$$\psi^\mathrm{T} = (\psi_\mathrm{R},\ \psi_\mathrm{L}), \quad j_\mathrm{R} \equiv \frac{1}{2}\left(j^0 + j^1\right), \quad j_\mathrm{L} \equiv \frac{1}{2}\left(j^0 - j^1\right). \tag{7}$$

From eq.(6), the current j^μ and its dual (axial) current $\widetilde{j}^\mu = \epsilon^{\mu\nu} j_\nu$ is conserved,

$$\partial_\mu j^\mu = 0, \quad \partial_\mu \widetilde{j}^\mu = 0. \tag{8}$$

The Thirring model is exactly solvable. Thirring [1] constructed the eigenstates while Klaiber [3] found the operator solution. On the other hand, Nakanishi [4] described the quantum operator solution of eq.(6) in terms of the free massless bosonic field $\varphi = \varphi_\mathrm{R}(x^-) + \varphi_\mathrm{L}(x^+)$ as

$$\psi(x) = \frac{Z}{\sqrt{2\pi}} \begin{pmatrix} :e^{\mathrm{i}\, s\varphi_\mathrm{R} - \mathrm{i}\, \bar{s}\varphi_\mathrm{L}}: \\ :e^{\mathrm{i}\, \bar{s}\varphi_\mathrm{R} - \mathrm{i}\, s\varphi_\mathrm{L}}: \end{pmatrix}, \tag{9}$$

where $s, \bar{s}$ are constant parameters and Z is a normalization factor. The free bosonic field satisfies

$$\partial_\mu \partial^\mu \varphi = 0 \tag{10}$$

and we can regularize [13] as

$$\left[\varphi_\mathrm{R}^\downarrow(x^-), \varphi_\mathrm{R}^\uparrow(y^-)\right] = -\frac{1}{4\pi} \ln \mathrm{i}(x^- - y^- - \mathrm{i}0), \tag{11}$$

$$\left[\varphi_\mathrm{L}^\downarrow(x^+), \varphi_\mathrm{L}^\uparrow(y^+)\right] = -\frac{1}{4\pi} \ln \mathrm{i}(x^+ - y^+ - \mathrm{i}0), \tag{12}$$

where $\varphi_{\mathrm{R,L}}^\downarrow$ and $\varphi_{\mathrm{R,L}}^\uparrow$ are the positive and the negative frequency part respectively. Therefore, we have the Operator Product Expansion (OPE) of the Thirring operator,

$$\begin{aligned} \psi_\mathrm{R}(x)\psi_\mathrm{R}(y) &= \frac{|Z|^2}{2\pi} \mathrm{i}^{\frac{s^2+\bar{s}^2}{4\pi}} (x^- - y^- - \mathrm{i}0)^{s^2/4\pi} (x^+ - y^+ - \mathrm{i}0)^{\bar{s}^2/4\pi} \\ &\quad \times :e^{\mathrm{i}\, s\varphi_\mathrm{R}(x) - \mathrm{i}\, \bar{s}\varphi_\mathrm{L}(x) + \mathrm{i}\, s\varphi_\mathrm{R}(y) - \mathrm{i}\, \bar{s}\varphi_\mathrm{L}(y)}:, \end{aligned} \tag{13}$$

$$\begin{aligned} \psi_\mathrm{R}^\dagger(y)\psi_\mathrm{R}(x) &= \frac{|Z|^2}{2\pi} \mathrm{i}^{-\frac{s^2+\bar{s}^2}{4\pi}} (y^- - x^- - \mathrm{i}0)^{-s^2/4\pi} (y^+ - x^+ - \mathrm{i}0)^{-\bar{s}^2/4\pi} \\ &\quad \times :e^{-\mathrm{i}\, s\varphi_\mathrm{R}(y) + \mathrm{i}\, \bar{s}\varphi_\mathrm{L}(y) + \mathrm{i}\, s\varphi_\mathrm{R}(x) - \mathrm{i}\, \bar{s}\varphi_\mathrm{L}(x)}: \end{aligned} \tag{14}$$

and so on. We have the similar relation for $\psi_{\rm L}$ if $\bar{s} \leftrightarrow s$. For the massless Dirac field case ($g=0$), we find $s=2\sqrt{\pi}$ and $\bar{s}=0$.

To solve the model, we must determine the parameters $s, \bar{s}$. The first condition of s and $\bar{s}$ is given by the aniti-commutativity of ψ and we have

$$\frac{s^2-\bar{s}^2}{4\pi}=1. \tag{15}$$

3 Current regularization

Next, we insert the operator solution into the field equation eq.(6). To do this, we propose the following current definition,

$$\begin{aligned} j^\mu(x) &= \frac{1}{2}\Big[\bar{\psi}(x+\varepsilon)\gamma^\mu\psi(x)+\bar{\psi}(x+\widetilde{\varepsilon})\gamma^\mu\psi(x)\Big] \\ &\quad -\frac{\sigma}{2}\alpha^\mu{}_\nu\Big[\bar{\psi}(x+\varepsilon)\gamma^\nu\psi(x)-\bar{\psi}(x+\widetilde{\varepsilon})\gamma^\nu\psi(x)\Big], \end{aligned} \tag{16}$$

where $\varepsilon(\widetilde{\varepsilon})$ is an infinitesimal timelike (spacelike) vector and σ is a parameter and $\alpha^0{}_0=-\alpha^1{}_1=1$, $\alpha^0{}_1=\alpha^1{}_0=0$. Here, we get timelike vector ε^μ close to zero with $\varepsilon^1\to 0$ firstly, whereas the spacelike vector $\widetilde{\varepsilon}$ is done in an opposite way. In our formulation, the current is written by

$$j_{\rm R}=-\frac{s-\sigma\bar{s}}{2\pi}\partial_-\varphi_{\rm R}, \quad j_{\rm L}=\frac{s-\sigma\bar{s}}{2\pi}\partial_+\varphi_{\rm L}. \tag{17}$$

Therefore, the operator solution (9) is valid if

$$\bar{s}=\frac{g}{2\pi}(s-\sigma\bar{s}). \tag{18}$$

Finally, we have the equations which must be satisfied by the parameters of the solution

$$\frac{s^2-\bar{s}^2}{4\pi}=1, \quad \bar{s}=\frac{g}{2\pi}(s-\sigma\bar{s}). \tag{19}$$

First, we consider $\sigma=0$ and $\sigma=1$ case. The second equation becomes

$$\bar{s}=\frac{g_{\sigma=0}}{2\pi}s, \quad \bar{s}=\frac{g_{\sigma=1}}{2\pi}(s-\bar{s}). \tag{20}$$

They can identify with the map,

$$g_{\sigma=1}=\frac{g_{\sigma=0}}{1-g_{\sigma=0}/2\pi}. \tag{21}$$

This is nothing but the relation between the coupling constant of the Schwinger definition and that of the Johnson definition. This is consistent with Klaiber's result [3]. Therefore, $\sigma=1$ corresponds to Schwinger's current definition and $\sigma=0$ is Johnson's one in our formulation.

We can also calculate the commutation rules between the current and the spinor field ψ,

$$\left[\psi(x^1,t),\ j^0(y^1,t)\right] = \frac{(s-\sigma\bar{s})(s+\bar{s})}{4\pi}\delta(x^1-y^1)\psi(x) \tag{22}$$

and

$$\left[\psi(x^1,t),\ j^1(y^1,t)\right] = \frac{(s-\sigma\bar{s})(s-\bar{s})}{4\pi}\delta(x^1-y^1)\gamma^5\psi(x). \tag{23}$$

4 Comparison with other formulations

Dell'Antonio, Frishman and Zwanziger [14] analyzed the Thirring model in a different way. They consider the commutation relations of the current, the Schwinger term. We can identify their result with

$$a = \frac{(s-\sigma\bar{s})(s+\bar{s})}{4\pi}, \quad \bar{a} = \frac{(s-\sigma\bar{s})(s-\bar{s})}{4\pi}, \quad c = \left(\frac{s-\sigma\bar{s}}{2\pi}\right)^2, \tag{24}$$

where a, $\bar{a}$ and c are parameters in their formulation (Note that in [14], $\epsilon_{\mu\nu}$ is defined by $\epsilon_{10}=1$). It is easy to check the consistency condition, eq.(6.1) in [14],

$$a - \bar{a} = gc. \tag{25}$$

c is written in terms of the coupling constant g as

$$1/c = \pi\left[1+\frac{g}{2\pi}(\sigma-1)\right]\left[1+\frac{g}{2\pi}(\sigma+1)\right]. \tag{26}$$

If $\sigma = 0$, it becomes eq.(6.3) in [14]. Therefore, our result perfectly agrees with Dell'Antonio et al. and the parameter of the current commutation relation is determined by the coupling constant and the parameter σ appeared in the current definition.

Taguchi, Tanaka and Yamamoto [15] consider the Thirring model with the Tomonaga-Schwinger equation. They consider the deformed hamiltonian and calculate the commutation relations between the current and the spinor field. In this case, we have eq.(22) and (23) in a similar way.

It is well-known that the Thirring model is $c=1$ (c is the central charge) Conformal Field Theory (CFT) [16]. Klassen and Melzer [17] argued that the Thirring model is equivalent to the *fermionic* Gaussian CFT. They show the relation between the compactification radius of the fermionic Gaussian CFT and the Thirring coupling constant. We give their result with $\sigma=0$ (Johnson current). More generally, the compactification radius R is written by

$$R = \frac{1}{\sqrt{\left[1+\frac{g}{2\pi}(\sigma-1)\right]\left[1+\frac{g}{2\pi}(\sigma+1)\right]}}\left[1 \pm \frac{g}{2\pi}\sqrt{1-\frac{4\pi}{g}\sigma-\sigma^2}\,\right]. \tag{27}$$

5 Conclusion

We have presented the generalization of the current regularization in the Thirring model. The definition of the current is complicated, but it becomes simple when it is expressed in terms of the free massless bosonic field. The present description is consistent with known results of the Thirring model. The present formulation is simpler than Klaiber's formulation. Klaiber defines the commutator of the current j^μ and the field ψ, and makes the anzatz about the current while we employ the operator solution which is given by Nakanishi [4]. The solution is written in terms of the *free massless bosonic field*, and thus we can easily evaluate various quantities. This is the main difference between Klaiber's treatment and ours. Further, we obtain the general formula for arbitrary current regularization.

The short distance behavior of the Thirring model is more complicated than the free massless Dirac field. For the Dirac field, the limiting procedures of $\varepsilon \to 0$ and $\widetilde{\varepsilon} \to 0$ are the same. On the other hand, they are different for the Thirring field. This is the consequence of the fact that the Dirac field ($\bar{s} = 0$) is written in terms of the bosonic field φ_{R} and φ_{L} separately in contrast with the Thirring field.

In the present description, we introduce a new parameter σ in our current definition. This current becomes Lorentz covariant limiting operator, $j^\mu \sim \partial^\mu \widetilde{\varphi}$. Here, $\widetilde{\varphi}$ is the dual massless field of φ[13]. We obtain the Johnson current for $\sigma = 0$ and the Schwinger current for $\sigma = 1$. Note that, for $\sigma = 1$, the current definition is given by

$$j^0(x) = \bar{\psi}(x + \widetilde{\varepsilon})\gamma^0\psi(x), \qquad j^1(x) = \bar{\psi}(x + \varepsilon)\gamma^1\psi(x). \tag{28}$$

Therefore, in our formulation, j^0 is defined by limiting from spacelike direction while j^1 is defined by limiting from timelike direction. This is in contrast with the original Schwinger's definition [10] which is defined by the spacelike separation only. However, we can show that j^0 and j^1 commute with each other for the Thirring case if we adopt the original prescription. On the other hand, in our formulation, the Schwinger term [18, 10] appear as

$$\big[j_0(x^1, t),\ j_1(y^1, t)\big] = \mathrm{i}\, c\, \delta'(x^1 - y^1), \tag{29}$$

where c is given by eq.(26). Accordingly, the current must be defined by limiting from both spacelike and timelike direction in the Thirring model. Note that the parameter σ determines the current algebra. Unfortunately, we do not understand the physical meaning of the parameter σ yet.

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References

[1] W. Thirring, Ann. Phys. (N.Y) **3** (1958) 91.
[2] C.R. Hagen, Nuovo Cimento **51B** (1967) 169.
[3] B. Klaiber, in Lectures in Theoretical Physics, 1967, eds. A. Barut and W. Britten (Gordon and Breach, NY, 1968), p141.
[4] N. Nakanishi, Prog. Theor. Phys. **57** (1977), 580, 1025
[5] *Bosonization*, edited by M. Stone (World Scientific, 1994).
[6] A. Ogura, Ph. D thesis, Nihon University (1994).
[7] K. Wilson, Phys. Rev. **179** (1969) 1499.
[8] L.P. Kadanoff, Phys. Rev. Lett. **23** (1969) 1430.
[9] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.
[10] J. Schwinger, Phys. Rev. Lett. **3** (1959) 296.
[11] K. Johnson, Nuovo Cimento, **20** (1961) 773.
[12] T. Fujita, T. Kake and H. Takahashi, Ann. Phys. **282** (2000) 100, hep-th/9906120.
[13] E. Abdalla, M.C.B. Abdalla and K.D. Rothe, *Non-perturbative methods in 2 Dimensional Quantum Field Theory*, (World Scientific, 1991).
[14] G.F. Dell'Antonio, Y. Frishman and D. Zwanziger, Phys. Rev. **D6** (1972) 988.
[15] Y. Taguchi, A. Tanaka and K. Yamamoto, Prog. Theor. Phys. **52** (1974) 1042.
[16] S. Ferrara, A.F. Grillo and R. Gatto, Nuovo Cimento **12A** (1972) 959.
[17] T.R. Klassen and E. Melzer, Int. J. Mod. Phys. **A8** (1993) 4131.
[18] T. Goto and T. Imamura, Prog. Theor. Phys. **14** (1955) 396.